Evaluation of Self-Intersecting Wilson Loop in the Stochastic Vacuum Model

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Abstract

A Wilson loop is evaluated within the stochastic vacuum model for the case when the respective contour is self-intersecting and its size does not exceed the correlation length of the vacuum. The result has the form of a certain functional of the tensor area. It is similar to that for the non-self-intersecting loop only when the contour is a plane one. Even for such a contour, the obtained expression depends on the ratio of two functions parametrizing the bilocal field strength correlator taken at the origin, which is not so for the case of non-self-intersecting contour.

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The small non-self-intersecting Wilson loop in QCD [1],

$$\langle W(C)\rangle \equiv \frac{1}{N_c}\left\langle \mathrm{tr}\,\mathcal{P}\,\exp\left(ig\oint dx_\mu A_\mu^a T^a\right)\right\rangle,$$

has been evaluated long time ago. First, it followed indirectly from the respective quark-antiquark potential emerging due to the dipole interaction with the external colourelectric field [1] and then was found explicitly by performing the direct calculation in QCD [2]. The term "small" here means that the typical size of the loop does not exceed the correlation length of QCD vacuum, T_g, *i.e.* the distance at which the bilocal field strength correlator in stochastic vacuum model [2, 3] decreases. This length has been measured in the lattice experiments [4, 5] (see Refs. [6, 7] for reviews) with the result $T_g \simeq 0.34\,\mathrm{fm}$ for the realistic case of $SU(3)$ full QCD. As a consequence of smallness of the loop *w.r.t.* T_g, the field strength correlator of stochastic vacuum model can with a good accuracy be approximated during the evaluation of such a loop by gluonic condensate. Therefore for small loops the QCD vacuum is viewed as that of QCD sum rules [8] characterized by the infinite correlation length. However, self-intersecting loops are also of a great importance for QCD, since these are those loops at which loop equations [9] are nontrivial. In particular, in 2D QCD such loops have been comprehensively studied in Ref. [10] (see Ref. [11] for a review). In the present letter, by combining stochastic vacuum model with the loop space approach we shall evaluate small self-intersecting Wilson loop in 4D QCD with arbitrary number of colours. As we shall eventually see, the resulting expression differs significantly from that for a non-self-intersecting loop.

The idea we are going to employ is based on the possibility to represent the loop-space Laplacian [12]

$$\Delta \equiv \int\limits_0^1 d\sigma \int\limits_{\sigma-0}^{\sigma+0} d\sigma' \frac{\delta^2}{\delta x_\mu(\sigma')\delta x_\mu(\sigma)} \tag{1}$$

standing on the L.H.S. of the loop equations in the following form [13]:

$$\Delta = \oint dx_\mu(\sigma)\,\mathsf{v.p.}\int d\sigma' \dot{x}_\nu(\sigma') \frac{\delta^2}{\delta\sigma_{\lambda\mu}(x(\sigma))\delta\sigma_{\nu\lambda}(x(\sigma'))}, \tag{2}$$

where $\mathsf{v.p.}\int d\sigma' \equiv \int\limits_0^{\sigma-0} d\sigma' + \int\limits_{\sigma+0}^1 d\sigma'$. It is worth mentioning once more that as it follows from the loop equations, the result of the action of the loop-space Laplacian onto the Wilson loop is nonvanishing only provided that this loop is self-intersecting. By virtue of Eq. (2), we get for such a loop the following equation:

$$\Delta\langle W(C)\rangle = -\frac{g^2}{N_c}\oint dx_\mu(\sigma)\,\mathsf{v.p.}\int d\sigma' \dot{x}_\nu(\sigma')\,\mathrm{tr}\,\langle F_{\lambda\mu}(x)\Phi(x,x')F_{\nu\lambda}(x')\Phi(x',x)\rangle\,. \tag{3}$$

Here, Δ is defined by Eq. (1), $x\equiv x(\sigma)$, $x'\equiv x(\sigma')$, $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]$ stands for the Yang-Mills field strength tensor, with $A_\mu \equiv A_\mu^a T^a$, and $\Phi(x,y) \equiv \frac{1}{N_c}\mathcal{P}\exp\left(ig\int\limits_y^x A_\mu(u)du_\mu\right)$ is a parallel transporter factor along the respective part of the contour C. We can now take into

[1] From now on, we shall for brevity call by self-intersecting loop the loop defined at a self-intersecting contour and imply by the size of the loop the size of this contour. Besides that, all the investigations will be performed in the Euclidean space-time.

account the fact that the characteristic size of the countour C under consideration does not exceed T_g [2]. This enables us to use for the field strength correlator standing on the R.H.S. of Eq. (3) the expression known from the stochastic vacuum model [2, 3]. In fact, for such a small contour joining the points x and x' the result for the bilocal field strength correlator suggested by this model is independent of the form of the contour and reads

$$\frac{g^2}{2}\left\langle F_{\mu\nu}(x)\Phi(x,x')F_{\lambda\rho}(x')\Phi(x',x)\right\rangle = \frac{\hat{1}_{N_c\times N_c}}{N_c}\Big\{(\delta_{\mu\lambda}\delta_{\nu\rho}-\delta_{\mu\rho}\delta_{\nu\lambda})\mathcal{D}(x-x')+$$

$$+\frac{1}{2}\left[\partial_\mu^x((x-x')_\lambda\delta_{\nu\rho}-(x-x')_\rho\delta_{\nu\lambda})+\partial_\nu^x((x-x')_\rho\delta_{\mu\lambda}-(x-x')_\lambda\delta_{\mu\rho})\right]\mathcal{D}_1(x-x')\Big\}. \qquad (4)$$

In what follows, we shall consider only nonperturbative parts of the functions $\mathcal{D}$ and $\mathcal{D}_1$, which were measured on the lattice in Refs. [4, 5, 6, 7]. That is because the perturbative contributions to these functions, relevant to the UV divergencies [14], yield a renormalization factor, which cancels out with the same factor appearing on the L.H.S. of Eq. (3) during the direct renormalization of the Wilson loop [3]. Therefore, from now on we shall deal with the renormalized Wilson loop as well as the renormalized charge g. Owing to Eq. (4), Eq. (3) takes the form $\Delta W[x]=J[x]$, where $W[x]\equiv\langle W(C)\rangle$ and $J[x]=\oint dx_\mu(\sigma)\,\mathsf{v.p.}\int d\sigma'\dot{x}_\mu(\sigma')F(x-x')$ with

$$F(x)\equiv\frac{1}{N_c}\left[6\mathcal{D}(x)+4\mathcal{D}_1(x)+x_\mu\partial_\mu\mathcal{D}_1(x)\right]. \qquad (5)$$

The above equation for the Wilson loop can be solved by virtue of the method of inversion of the loop-space Laplacian proposed in Ref. [16] [4]. The idea of this method is to replace the original loop-space Laplacian (1) by the smeared one $\Delta^{(G)}=\int\limits_0^1 d\sigma\int\limits_0^1 d\sigma' G(\sigma-\sigma')\frac{\delta^2}{\delta x_\mu(\sigma')\delta x_\mu(\sigma)}$, where $G(\sigma-\sigma')$ is a certain smearing function. Such a smeared Laplacian can be inverted, which yields

$$W[x]=1-\frac{1}{2}\int\limits_0^\infty dA\left(\left\langle J\left[x+\sqrt{A}\xi\right]\right\rangle_\xi^{(G)}-\left\langle J\left[\sqrt{A}\xi\right]\right\rangle_\xi^{(G)}\right). \qquad (6)$$

Here,

$$\langle\mathcal{O}[\xi]\rangle_\xi^{(G)}=\frac{\int\limits_{\xi(0)=\xi(1)}\mathcal{D}\xi\mathrm{e}^{-S}\mathcal{O}[\xi]}{\int\limits_{\xi(0)=\xi(1)}\mathcal{D}\xi\mathrm{e}^{-S}}$$

is the Gaussian average over loops with the action $S=\frac{1}{2}\int\limits_0^1 d\sigma\int\limits_0^1 d\sigma'\xi(\sigma)G^{-1}(\sigma-\sigma')\xi(\sigma')$, where $G^{-1}(\sigma-\sigma')$ denotes the inverse operator. Following Ref. [16], we shall choose $G(\sigma-\sigma')$ in the form $G(\sigma-\sigma')=\mathrm{e}^{-|\sigma-\sigma'|/\varepsilon}$, $\varepsilon\ll 1$, after which the above action becomes local: $S=\frac{1}{4}\int\limits_0^1 d\sigma\left(\varepsilon\dot{\xi}^2(\sigma)+\frac{1}{\varepsilon}\xi^2(\sigma)\right)$.

[2] More rigorously, this means that the area of the minimal surface spanned by C is not larger than T_g^2 and that C does not contain appendix-shaped pieces of the vanishing area, but large length.

[3] The multiplicative renormalizability of self-intersecting Wilson loop has been proved in Ref. [15].

[4] This method has been successfully applied in Ref. [17] to the solution of the Cauchy problem for the loop equation in turbulence.

Next, the averages on the R.H.S. of Eq. (6) are similar to those which one carries out during the reproduction of the one-gluon-exchange diagram within the same method when it is applied to the usual loop equation in the large-N_c limit [16]. In particular, this follows from the fact that the integral operator present in $J[x]$ accidentally has the same form as the one standing on the R.H.S. of the large-N_c loop equation. In the limit $\varepsilon \to 0$, the only nonvanishing contribution appears in the first of the two averages on the R.H.S. of Eq. (6) and reads

$$W[x] = 1 - \frac{1}{2}\int\limits_0^\infty dA \int\limits_0^1 d\sigma \int\limits_0^1 d\sigma'(1-G(\sigma-\sigma'))\dot{x}_\mu(\sigma)\dot{x}_\mu(\sigma')\times$$

$$\times \int \frac{d^4p}{(2\pi)^4} \exp\left[-Ap^2(1-G(\sigma-\sigma'))+ip(x(\sigma)-x(\sigma'))\right]\tilde{F}(p). \tag{7}$$

Here, the factor $\mathrm{e}^{-Ap^2(1-G(\sigma-\sigma'))}$ is the result of the average $\left\langle \mathrm{e}^{i\sqrt{A}p(\xi(\sigma)-\xi(\sigma'))}\right\rangle_\xi^{(G)}$, and $\tilde{F}(p) \equiv \int d^4x \mathrm{e}^{-ipx}F(x)$ is the Fourier image of the function $F(x)$. Next, in Eq. (7), the factor $(1-G(\sigma-\sigma'))$ in the preexponent was introduced in order to make out of the full integral over σ' the principal-value one in the limit $\varepsilon \to 0$. This factor disappears upon the introduction instead of A the new integration variable $\alpha = \Lambda^{-2} + 2A(1-G(\sigma-\sigma'))$, where Λ stands for the UV momentum cutoff. Sending Λ to infinity we arrive at the following expression:

$$W[x] = 1 - \frac{1}{2}\oint dx_\mu \oint dx'_\mu \int \frac{d^4p}{(2\pi)^4}\frac{\mathrm{e}^{ip(x-x')}}{p^2}\tilde{F}(p) =$$

$$= 1 - \frac{1}{8\pi^2}\oint dx_\mu \oint dx'_\mu \int d^4z \frac{F(z)}{(z-(x-x'))^2}. \tag{8}$$

The infinite integral (*i.e.* the integral over z or p) can be calculated in the small-loop case under study. In this limit, one can replace $\mathcal{D}(x)$ and $\mathcal{D}_1(x)$ by their values at the origin, which according to Eq. (4) are related to each other as

$$\mathcal{D}(0) + \mathcal{D}_1(0) = \frac{g^2}{24}\,\mathrm{tr}\,\left\langle F_{\mu\nu}^2(0)\right\rangle. \tag{9}$$

On the other hand, according to the lattice measurements [4, 5, 6, 7], $\mathcal{D}_1(0) = \alpha\mathcal{D}(0)$, where $\alpha \simeq 0.2 \pm 0.1$ (see also Ref. [18] for the discussion of this value of α). This yields the following approximate constant value of the function $F(z)$: $F(z) \simeq \frac{3+2\alpha}{1+\alpha}\frac{g^2}{12N_c}\,\mathrm{tr}\,\left\langle F_{\mu\nu}^2(0)\right\rangle \equiv \mathcal{C}$. The remaining infinite integral can easily be calculated in the limit $T_g \gg |x-x'|$ under study *e.g.* from the first equality on the R.H.S. of Eq. (8). We have

$$\int \frac{d^4p}{(2\pi)^4}\frac{\mathrm{e}^{ip\lambda}}{p^2}\tilde{F}(p) = \mathcal{C}\int d^4p\delta(p)\frac{\mathrm{e}^{ip\lambda}}{p^2} \simeq \frac{\mathcal{C}T_g^4}{16\pi^2}\int\limits_0^\infty ds \int d^4p \mathrm{e}^{ip\lambda - p^2\left(\frac{T_g^2}{4}+s\right)} = \frac{\mathcal{C}T_g^4}{4\lambda^2}\left(1-\mathrm{e}^{-\frac{\lambda^2}{T_g^2}}\right),$$

where $\lambda \equiv x - x'$, and thus

$$W[x] = 1 - \frac{\mathcal{C}T_g^4}{8}\oint dx_\mu \oint dx'_\mu \frac{1}{(x-x')^2}\left[1-\mathrm{e}^{-\frac{(x-x')^2}{T_g^2}}\right].$$

Expanding the exponential in this formula we finally arrive at the following leading nontrivial contribution to the Wilson loop:

$$\langle W(C)\rangle \simeq 1-\frac{3+2\alpha}{1+\alpha}\frac{g^2}{96N_c}\,\mathrm{tr}\,\left\langle F_{\mu\nu}^2(0)\right\rangle \Sigma_{\mu\nu}^2, \tag{10}$$

where $\Sigma_{\mu\nu}\equiv\oint dx_\mu x_\nu$ is the tensor area corresponding to the contour C. Note that T_g dropped out from this leading term, as it could be expected from the beginning, since $1/T_g$ was considered as an IR cutoff.

The obtained expression for the Wilson loop is now worth to be compared with the respective expression for the small non-self-intersecting Wilson loop. In that case owing to the non-Abelian Stokes theorem and Eq. (4) one has

$$\langle W(C)\rangle \simeq \frac{1}{N_c}\,\mathrm{tr}\,\exp\left(-\frac{g^2}{8}\int\limits_{\Sigma_{\min}[C]} d\sigma_{\mu\nu}(x)\int\limits_{\Sigma_{\min}[C]} d\sigma_{\lambda\rho}(x')\left\langle F_{\mu\nu}(x)\Phi(x,x')F_{\lambda\rho}(x')\Phi(x',x)\right\rangle\right)=$$

$$=\exp\left(-\frac{g^2}{48N_c}\,\mathrm{tr}\,\left\langle F_{\mu\nu}^2(0)\right\rangle\int\limits_{\Sigma_{\min}[C]} d\sigma_{\mu\nu}(x)\int\limits_{\Sigma_{\min}[C]} d\sigma_{\mu\nu}(x')\right).$$

Here, $x\equiv x(\xi)$ is the vector parametrizing the surface of the minimal area spanned by the contour C, $\Sigma_{\min}[C]$, and $\xi=(\xi^1,\xi^2)$ stands for the 2D-coordinate. Next, $d\sigma_{\mu\nu}(x)=\sqrt{g(\xi)}t_{\mu\nu}(\xi)d^2\xi$, where $g(\xi)$ is the determinant of the induced metric tensor $g^{ab}(\xi)=(\partial^a x_\mu(\xi))(\partial^b x_\mu(\xi))$ and $t_{\mu\nu}(\xi)=\varepsilon^{ab}\left(\partial_a x_\mu(\xi)\right)\left(\partial_b x_\nu(\xi)\right)/\sqrt{g(\xi)}$ is the extrinsic curvature tensor. Since the contour C under discussion is very small (and consequently the same is $\Sigma_{\min}[C]$), the points x and x' are located very closely to each other, and therefore $t_{\mu\nu}(\xi)t_{\mu\nu}(\xi')\simeq t_{\mu\nu}^2(\xi)=2$. Finally, taking into account that $\int d^2\xi\sqrt{g(\xi)}=$ Area of $\Sigma_{\min}[C]\equiv S_{\min}$, we obtain [2] $\langle W(C)\rangle\simeq 1-\frac{g^2}{24N_c}\,\mathrm{tr}\,\left\langle F_{\mu\nu}^2(0)\right\rangle S_{\min}^2$. For a *plane* contour, this expression has the form similar to our Eq. (10), since for such a contour

$$S_{\min}^2=\frac{1}{2}\Sigma_{\mu\nu}^2. \tag{11}$$

In this case, the main difference between these two expressions stems from the α-dependence of Eq. (10). This dependence is due to the fact that the functions $\mathcal{D}$ and $\mathcal{D}_1$ contribute to the self-intersecting loop in the nontrivial combination (5), whereas the non-self-intersecting loop depends only on their sum at the origin, expressible owing to Eq. (9) via the condensate alone. Another obvious difference of the two expressions for Wilson loops is that the tensor area for self-intersecting countour can be vanishingly small even for a very large contour (although we do not consider such contours) and even vanish completely for the eight-shaped contour with equal petals, whereas for a non-self-intersecting contour it could vanish only together with the contour itself. Moreover, in this respect it is worth pointing out once more that the comparison of the results for self-intersecting and non-self-intersecting Wilson loops is only possible for plane contours, since for non-plane ones Eq. (11) is not valid.

In conclusion, by making use of the method of inversion of the loop-space Laplacian, we have restored a small self-intersecting Wilson loop from the bilocal field strength correlator of stochastic vacuum model. There turned out to be two main differences of the obtained result (10) from that of the non-self-intersecting loop. Firstly, Eq. (10) depends on the tensor area of the contour, rather than on the area of the minimal surface, and can therefore vanish for some class of contours (*e.g.* for plane eight-shaped contours with equal petals). Secondly, the obtained result depends on the ratio of nonperturbative parts of the functions $\mathcal{D}_1(x)$ and $\mathcal{D}(x)$ at the origin, which is not the case for a non-self-intersecting contour. However, for plane contours, the functional form of the obtained result coincides with that of a small non-self-intersecting loop when the latter one is expressed in terms of the tensor area.

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